AN ATTEMPT TO CONSTRUCT A THEORY OF FRACTURE FOR BRITTLE MATERIALS, BASED ON GRIFFITH'S CRITERION

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In Griffith's paper [1] a theory was proposed, explaining the rupture of brittle materials by uniaxial tension in the presence of microscopic cracks. The basic idea of Griffith's theory concludes, that on the surface of a solid body forces of tension act, analogous to forces acting on the surface of a fluid, and that the decrease in the potential energy W^* of the body caused by the formation of a crack of length $2l$ is compensated by the surface energy of the crack. In order for a given crack to grow, it is necessary that the change of free energy of the body W^*-U must not increase with the increase of the crack dimension, i.e.

$$\frac{\partial}{\partial l}(W^* - U) = 0 \qquad (0.1)$$

From relation (0.1) a critical parameter for the equilibrium state is found.

The surface energy of the crack

$$U = 4lT_0 \qquad (0.2)$$

where $2T_0$ is the energy required for the formation of one length unit of the crack.

The surface tension T_0, for sufficiently general assumptions, may be considered constant for a given material.

Griffith obtained the relation determining the critical value of fracture stress for uniaxial tension of an infinite plate with a line crack of length $2l$, by forces perpendicular to the line of the crack, in the form

$$p_0 = (2ET_0 / \pi l)^{1/2}, \qquad p_0 = (2ET_0 / \pi l\,(1 - \nu^2))^{1/2} \qquad (0.3)$$

for the conditions of a plane state of stress and of plane strain, respectively. Here E is Young's modulus, ν is Poisson's coefficient.

In the present analysis we attempt to make use of an idea of Griffith for the construction of a theory for the failure of brittle solids for the cases when the assumption of either a plane state of stress or of plane strain is realized.

The theory of fracture here is undestood in the sense, usually taken in the science of the resistance of materials, namely: to find a function $F_1(\sigma_1, \sigma_2, \sigma_3)$ such that, in order that the condition $F(\sigma_1, \sigma_2, \sigma_3) = C$ be fulfilled, a failure of the metrial must have occurred.

In the present analysis for the construction of a theory we consider the following hypotheses:

1) The crack propagates, remaining rectilinear.

2) The crack is located perpendicular to the surface of the plate.

3) The quantities T_0 and l are constants of the material.

For the calculation of the potential energy of deformation it is necessary to overcome the sometimes significant difficulty of calculation. In Section 1 equations are derived that allow one to evaluate the potential energy of an infinite plate with a cut of arbitrary form in the case of an isotropic state of stress at infinity, provided that the coefficients of the functions of Muskhelishvili near an infinitely distant point are known.

In Sections 2 and 3 two cases of crack propagation, that of tension (Section 2) and that of shear (Section 3) are considered.

1. Derivation of equations for potential energy. The potential energy of a plate may be computed by Formula

$$W = \frac{1}{2}\iint_D \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] dx\,dy \tag{1.1}$$

Here, as in the sequel, the notation of [2] is assumed; the thickness of the plate is assumed equal to unity; the integration in (1.1) is carried out over the area of the plate. Integrating by parts and taking into account that

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

we obtain

$$W = {}^1/_2 \oint [(\sigma_x u + \tau_{xy} v)\cos\theta + (\sigma_y v + \tau_{xy} u)\sin\theta]\,ds \tag{1.2}$$

or

$$W = {}^1/_2 \mathrm{Re} \oint (u + iv)\,[(\sigma_x - i\tau_{xy})\cos\theta + (\tau_{xy} - i\sigma_y)\sin\theta]\,ds \tag{1.3}$$

where Re denotes the real part of the complex expression.

It is assumed that the work of the forces is accomplished only on the exterior contour.

As is well-known [2], the state of stress and deformation of an elastic, isotropic medium in a plane problem is determined by two analytic functions $\varphi(z)$ and $\psi(z)$, with which the components of stress and displacement are associated by the relations

$$\begin{gathered} \sigma_x + \sigma_y = 2\,[\varphi'(z) + \overline{\varphi'(z)}] \qquad (z = x + iy) \\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2\,[\bar{z}\varphi''(\bar{z}) + \psi'(z)] \qquad \left(\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu}\right) \\ 2\mu\,(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \end{gathered} \tag{1.4}$$

where λ and μ are the Lame coefficients. From Equations (1.4) we have

$$\begin{gathered} \sigma_x - i\tau_{xy} = \varphi'(z) + \overline{\varphi'(z)} - \bar{z}\varphi''(z) - \psi'(z) \\ \tau_{xy} - i\sigma_y = -\,i\,[\varphi'(z) + \overline{\varphi'(z)} + \bar{z}\varphi''(z) + \psi'(z)] \end{gathered} \tag{1.5}$$

Taking into account Equations (1.5) and the last equation from (1.4), we write Equations (1.3) in the form

$$W = {}^1/_4\mu^{-1}\,\mathrm{Re}\oint [\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]\,\{[\varphi'(z) + \overline{\varphi'(z)}]\,(\cos\theta - {} \\ {} - i\sin\theta) - [\bar{z}\varphi''(z) + \psi'(z)]\,(\cos\theta + i\sin\theta)\}\,ds \qquad (1.6)$$

If the integration is performed on the contour of a circle of radius R, then

$$\bar{z} = R^2 z^{-1}, \ (\cos\theta + i\sin\theta)\,ds = -\,idz \\ (\cos\theta - i\sin\theta)\,ds = -\,iR^2 z^{-2}dz \qquad (1.7)$$

If we use (1.7) in Equation (1.6), we obtain

$$W = \frac{1}{4\mu}\,\mathrm{Re}\oint [\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]\left\{\frac{R^2}{z}\varphi''(z) + \psi'(z) - {}\right. \\ \left. {} - \frac{R^2}{z^2}\,[\varphi''(z) + \overline{\varphi''(z)}]\right\} idz \qquad (1.8)$$

The potential energy of an infinite plate may be derived from the potential energy of a circular plate of radius R by means of passing to the limit as $R \to \infty$.

The solution to the problem of determining the state of stress in a circular plate of radius R with a crack of length $2l$ (and in general with an arbitrary configuration) may be obtained by means of a sequence of superposition of solutions for an infinite plate with a crack and for a continuous circular plate (Schwartz's algorithm).

Components of stress and displacement of a circular plate without a crack are determined by the functions

$$\varphi_1(z) = \Gamma z, \ \psi_1(z) = \Gamma' z, \quad (\Gamma = {}^1/_4(\sigma_1 + \sigma_2), \quad \Gamma' = -\,{}^1/_2(\sigma_1 + \sigma_2)\,e^{-2i\alpha}) \ (1.9)$$

Here σ_1, σ_2 are the values of principal stresses at infinity, α is the angle which the principal axis corresponding to σ_1 makes with the x-axis.

Functions $\varphi_1(z)$ and $\psi_1(z)$ cause certain stresses on the contour of the crack which must be removed by the introduction of functions

$$\varphi_2(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots, \qquad \psi_2(z) = \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots \quad (1.10)$$

Let l be the characteristic dimension of the hole. Using the conformal mapping $z_1 = z/l$, we reduce the problem to the consideration of an infinite plate with a slit of unit length. In this case the coefficients of the expansion of functions $\varphi(z_1)/l, \psi(z_1)/l$ in powers of z are dimensionless variables.

It follows that coefficients a_k, b_k contain factors l^{k+1}. This result may also be obtained from dimensional considerations.

In order to remove the displacements on the contours of the circle of radius R, caused by functions $\varphi_2(z)$ and $\psi_2(z)$, we introduce functions

$$\varphi_3(z) = A_1 z + A_2 z^2 + A_3 z^3 + \ldots \\ \psi_3(z) = B_1 z + B_2 z^2 + B_3 z^3 + \ldots \qquad (1.11)$$

It is required that functions $\varphi_2(z) + \varphi_3(z)$ and $\psi_2(z) + \psi_3(z)$ do not cause displacements on the contours of the circle, then retaining only terms containing l^2 (none higher), we have

$$A_1 = \frac{b_1}{(\varkappa - 1) R^2}, \quad A_2 = 0, \quad A_3 = -\frac{\bar{a}_1}{\varkappa R^4}, \quad A_4 = A_5 = \ldots = 0 \quad (1.12)$$

$$B_1 = \frac{\varkappa^2 + 3}{\varkappa R^2} \bar{a}_1, \qquad B_2 = B_3 = \ldots = 0$$

Thus, the unknown functions are (1.13)

$$\varphi(z) = (\Gamma + A_1) z + A_3 z^3 + \frac{a_1}{z} + \ldots, \qquad \psi(z) = (\Gamma' + B_1) z + \frac{b_1}{z} + \ldots$$

Substituting (1.13) in Equation (1.8) and using the theorem of residues for their integration, we obtain equation for the potential energy of the plate weakened by the crack

$$W = {}^1/_2 \pi \mu^{-1} \operatorname{Re} [2 (\varkappa - 1) \Gamma^2 R^2 + \Gamma' \bar{\Gamma}' R^2 + (\varkappa + 1) \bar{\Gamma}' \bar{a}_1 + (\varkappa - 1) \Gamma b_1 + 2 \Gamma \bar{b}_1] \quad (1.14)$$

The decrease of potential energy of the plate W^*, caused by the formation of a crack is

$$W^* = {}^1/_2 \pi \mu^{-1} \operatorname{Re} [(\varkappa + 1) \bar{\Gamma}' \bar{a}_1 + (\varkappa - 1) \Gamma b_1 + 2 \Gamma \bar{b}_1] \quad (1.15)$$

2. Biaxial tension of plate. We consider the tension of an isotropic elastic plate weakened by a crack of length $2l$. The contour of the crack is free of external stress, and the state of stress at infinity is represented by tensile stresses (p = const) uniformly distributed in a direction making an arbitrary angle α with the line of the crack; in the perpendicular direction $p_1 = ap$. The origin of coordinates is located at the crack center and the x-axis is directed along the crack (Fig.1). In this case functions $\Phi(z)$ and $\Omega(z)$, by which the components of stress and displacements may be expressed, have the form [2]

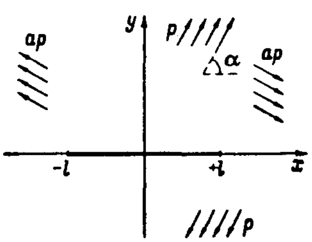

Fig. 1

$$\Phi(z) = \frac{2\Gamma + \bar{\Gamma}'}{2} \frac{z}{\sqrt{z^2 - l^2}} - \frac{1}{2} \bar{\Gamma}'$$

$$\Omega(z) = \frac{2\Gamma + \bar{\Gamma}'}{2} \frac{z}{\sqrt{z^2 - l^2}} + \frac{1}{2} \bar{\Gamma}' \quad (2.1)$$

where

$$\Omega(z) = \bar{\Phi}(z) + z \bar{\Phi}'(z) + \bar{\Psi}(z)$$

$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z) \quad (2.2)$$

Here Γ, Γ' are constants , they characterize the state of stress at infinity, and for the case under consideration, they have the values

$$\Gamma = {}^1/_4 p (1 + a), \qquad \Gamma' = -{}^1/_2 p (1 - a) e^{-2i\alpha} \quad (2.3)$$

Passing from functions $\Phi(z)$ and $\Psi(z)$ to functions $\varphi(z)$ and $\psi(z)$ and taking (2.3) into account, we find

$$\varphi(z) = \Gamma z + z^{-1} a_1 + \ldots \qquad \psi(z) = \Gamma'(z) + z^{-1} b_1 + \ldots \quad (2.4)$$

Here we denote

$$a_1 = -\,{}^1/_8 p l^2\,[1 + a - (1 - a)\, e^{2i\alpha}]$$

$$b_1 = -\,{}^1/_8 p l^2\,[2\,(1 + a) - (1 - a)\,(e^{2i\alpha} + e^{-2i\alpha})] \tag{2.5}$$

If we substitute the value of a_1 and b_1 in Equation (1.15) and use (2.3), we find the decrease in potential energy caused by the presence of the crack

$$W^* = \frac{\pi(\varkappa + 1)}{16\mu} p^2 l^2\,[1 + a^2 - (1 - a^2)\cos 2\alpha] \tag{2.6}$$

From condition (0.1) we determine the critical length of the crack

$$l = \frac{32\mu T_0}{\pi(\varkappa + 1)\, p^2\,[1 + a^2 - (1 - a^2)\cos 2\alpha]} \quad \left(l_1 = \frac{16\mu T_0}{\pi(\varkappa + 1)\, p^2}, \quad l_2 = \frac{16\mu T_0}{\pi(\varkappa + 1)\, p^2 a^2}\right) \tag{2.7}$$

Here in the parantheses are indicated, respectively, the minimal length of crack for $a < 1$ when $\alpha = \pi/2$ and for $a > 1$ when $\alpha = 0$.

Introducing the notation $\sigma_1 = p,\ \sigma_2 = ap$ and taking into account that $\mu = E\,/\,2\,(1 - \nu)$, and for the state of plane stress $\varkappa = (3 - \nu)\,/\,(1 + \nu)$, from Expressions (2.7) for l_1 and l_2 we obtain

$$\sigma_1 = \sigma_2 = \sqrt{2ET_0\,/\,\pi l} \tag{2.8}$$

For plane strain $\varkappa = 3 - 4\nu$ and Equation (2.10) has the form

$$\sigma_1 = \sigma_2 = \sqrt{2ET_0\,/\,\pi l\,(1 - \nu^2)} \tag{2.9}$$

3. Biaxial compression of a plate. We consider the compression of an elastic isotropic plate, weakened by a line crack of length $2l$. The width of the crack is so small that it may be neglected ("mathematical slit"). The contour of the crack is free of stress and the state of stress at infinity is represented by biaxial compressive stresses (p = const) uniformly distributed in the direction at an arbitrary angle α to the line of the crack, and $P_1 = ap$ perpendicular to p. The origin of coordinates is placed in the center of the crack, the x-axis is directed along the line of the crack (Fig.2). For certain hypotheses one may assume that with increased load the crack will be closed all along its edges. The force of friction along the edge of the crack will be neglected.

Boundary conditions on the contour of the crack are

$$\sigma_y^+ = \sigma_y^-, \quad \tau_{xy}^+ = \tau_{xy}^- = 0, \quad v^+ - v^- = 0 \qquad (-\,l < t < l) \tag{3.1}$$

The components of stress and displacement are given in terms of analytic functions $\Phi(z)$ and $\Omega(z)$ by the relations

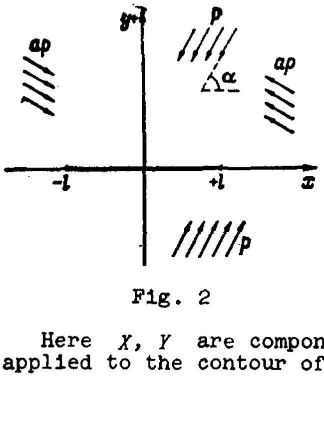

Fig. 2

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar z) - (z - \bar z)\,\overline{\Phi'(z)} \tag{3.2}$$

$$2\mu\,(u' + iv') = \varkappa\Phi(z) - \Omega(\bar z) - (z - \bar z)\,\overline{\Phi'(z)} \tag{3.3}$$

For sufficiently large $|z|$ functions $\Phi(z)$ and $\Omega(z)$ have the form

$$\Phi(z) = \Gamma - \frac{X + iY}{2\pi(1 + \varkappa)}\frac{1}{z} + O(z^{-2}),$$

$$\Omega(z) = \overline{\Gamma} + \overline{\Gamma}' + \frac{X + iY}{2\pi(1 + \varkappa)}\cdot\frac{1}{z} + O(z^{-2}) \tag{3.4}$$

Here X, Y are components of the principal external stress vector, applied to the contour of the crack

$$\Gamma = -{}^1/_4 p\,(1 + a), \qquad \Gamma' = {}^1/_2 p\,(1 - a)\,e^{-2i\alpha} \tag{3.5}$$

From Formulas (3.2) and (3.3)and from the quantities on the contour of the crack which are conjugate to them after passage to the boundary values, as well as using Equations (3.1) and certain transformations, we obtain four problems of linear relationship

$$\begin{gathered}
[\Phi(t) - \overline{\Phi}(t) - \Omega(t) + \overline{\Omega}(t)]^+ = [\Phi(t) - \overline{\Phi}(t) - \Omega(t) + \overline{\Omega}(t)]^- \\
[\varkappa\Phi(t) + \varkappa\overline{\Phi}(t) + \Omega(t) + \overline{\Omega}(t)]^+ = [\varkappa\Phi(t) + \varkappa\overline{\Phi}(t) + \Omega(t) + \overline{\Omega}(t)]^- \\
[\Phi(t) + \overline{\Phi}(t) - \Omega(t) - \overline{\Omega}(t)]^+ = [\Phi(t) + \overline{\Phi}(t) - \Omega(t) - \overline{\Omega}(t)]^- \\
[\Phi(t) - \overline{\Phi}(t) + \Omega(t) - \overline{\Omega}(t)]^+ = -[\Phi(t) - \overline{\Phi}(t) + \Omega(t) - \overline{\Omega}(t)]^-
\end{gathered} \tag{3.6}$$

Solving (3.6) with regard to (3.4), we obtain

$$\begin{gathered}
\Phi(z) = \frac{p}{4}\left[-(1 + a) + i\,(1 - a)\frac{z \sin 2\alpha}{\sqrt{z^2 - l^2}} - i\,(1 - a)\sin 2\alpha\right] \\
\Omega(z) = \frac{p}{4}\left[-(1 + a) + 2\,(1 - a)\cos 2\alpha + \frac{i\,(1 - a)\,z \sin 2\alpha}{\sqrt{z^2 - l^2}} + i(1 - a)\sin 2\alpha\right]
\end{gathered} \tag{3.7}$$

Making use of (2.2), we find from (3.7)

$$\begin{gathered}
\varphi(z) = {}^1/_4 p\,[-(1 + a)\,z - {}^1/_2 i l^2 (1 - a)\sin 2\alpha z^{-1} - \ldots] \\
\psi(z) = {}^1/_4 p\,[2\,(1 - a)(\cos 2\alpha - i \sin 2\alpha) - {}^1/_4 i l^4 (1 - a)\sin 2\alpha z^{-3} - \ldots]
\end{gathered} \tag{3.8}$$

It follows that

$$a_1 = -{}^1/_8 p l^2 i\,(1 - a)\sin 2\alpha, \qquad b_1 = 0 \tag{3.9}$$

Taking into account (3.9) and (3.5) we find from Equation (1.15)

$$W^* = \frac{\pi p^2 l^2 (\varkappa + 1)}{32\mu}(1 - a)^2 \sin^2 2\alpha \tag{3.10}$$

From relation (0.1) we obtain the critical crack length

$$l_* = \frac{64\mu T_0}{\pi\,(1 + \varkappa)\,p^2\,(1 - a)^2 \sin^2 2\alpha} \tag{3.11}$$

The minimum length of the crack occurs when $\alpha = \pi/4$

$$\min l = \frac{64\mu T_0}{\pi\,(1 + \varkappa)\,p^2\,(1 - a)^2} \tag{3.12}$$

Denoting $\sigma_1 = p$, $\sigma_2 = ap$ and substituting $\mu = E/2(1-\nu)$, $\varkappa = (3-\nu)/(1+\nu)$, from (3.12) we find the relation between the principal stresses σ_1 and σ_2 in the case of a state of plane stress

$$\sigma_1 - \sigma_2 = \pm 2\sqrt{2ET_0/\pi l} \tag{3.13}$$

For the case of plane deformation this relation is

$$\sigma_1 - \sigma_2 = \pm 2\sqrt{2ET_0/\pi l\,(1 - \nu^2)} \tag{3.14}$$

On the basis of Equations (2.8) and (3.13) we can show graphically the relation between the principal stresses σ_1 σ_2 for brittle fracture (Fig.3).

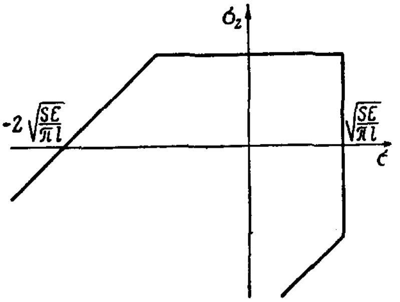

Fig. 3

In particular, for brittle fracture compressive stresses are twice as great as tensile. This result is very consistent with experimental data, obtained by the authors in test on facture specimens of brittle plastics. From the very formulation of Griffith's problem, it is clear that results obtained both by Griffith himself [1] and in the present article, are valid only for the case of the state of homogeneous stress.

It is completely obvious that the investigation of different cases of non-homogeneous states is necessary to be conducted separately for each case, and it is completely possible that the instant of collapse will be determined not only by the magnitude of the stresses but also by the stress gradients.

In his paper [3] Griffith undertook to attempt to construct a theory of fracture for the state of plane stress. Griffith proceeded from the solution to the problem in the theory of elasticity for the plane with an elliptical hole. On the boundary of the ellipse the state of stress was uniaxial,, therefore the maximal tensile stress was determined and compared with the critical stress, obtained earlier [1] for the nomogeneous uniaxial state.

In the expression for the maximal stress the values of the principal stresses at infinity and the angle of inclination of the ellipse were entered. Minimizing this expression for the angle (roughly, as in the present paper), Griffith in the end obtained the criterion for strength, whose application for the case of simple compression gives the fracture stress which is approximately eight times larger than the fracture stresses for tension.

The hypothesis formulated in this paper on the propagation of a crack in its plane only, should undoubtedly give excessive values for the fracture stresses in the nonsymmetrical cases.

Therefore, the results obtained in [3] seem to be incorrect.

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